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CRACK OPENING DISPLACEMENTS AND STRESS INTENSITY FACTORS CAUSED BY A CONCENTRATED LOAD OUTSIDE A CIRCULAR CRACK

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Abstract—This paper gives an evaluation of crack opening displacements and stress intensity factors in terms of elementary functions for the problem of a concentrated load outside a circular crack. The results are obtained for both transversely isotropic and purely isotropic cases. A particular case of a concentrated load at a point on the normal axis is discussed and compared with the previous analysis. Some results for stress intensity factors are presented in a graphical form.

INTRODUCTION

The concept of "weight functions" in two-dimensional elastic crack analysis was first introduced by Bueckner (1970). His weight functions are elastic displacement fields which equilibrate zero body forces and zero surface tractions but have a stronger singularity at the crack front than normally admissible displacement fields. Subsequent to Bueckner's analysis, Rice (1972) showed that weight functions could be evaluated by differentiating with respect to crack length known displacement solutions for two-dimensional crack problems. Rice (1972, 1989) has also laid the foundation for three-dimensional weight function theory based on displacement field variations cause by a first order variation in the position of a crack front. However, in their simplest interpretation, weight functions can be considered as the stress intensity factors around a crack front caused by an arbitrarily located concentrated force.

Since their inception, weight functions have played an important role in fracture mechanics and a great deal of effort has been aimed at evaluating weight functions for various crack geometries. In the present study the focus is specifically placed on closed form solutions to weight functions for three-dimensional geometries. Perhaps the first weight functions evaluated in closed form are the so-called "crack face weight functions" which are the stress intensity factors when the concentrated loading acts on the crack faces. Such solutions for half-plane, penny-shaped and circular external cracks in isotropic bodies can be extracted from the work of Galin (1961), Ufliand (1965), Tada *et al.* (1973) and Kassir and Sih (1975). A few additional solutions not present in these works can be found in more recent studies such as those of Meade and Keer (1984) and Fabrikant (1989). In previous investigations, loading was generally either symmetric or antisymmetric about the crack plane. Thus one can superpose solutions, say for symmetric normal loading and antisymmetric normal loading, to obtain the solution for concentrated normal loading on one crack face only.

Though many solutions exist for crack face weight functions for the crack shapes noted above, few closed form solutions have previously been given for general weight functions. The books by Tada *et al.* (1973) and Kassir and Sih (1975) generally summarized the known closed form results for penny-shaped cracks and circular external cracks when the point forces are located off the crack plane on the axis of symmetry. The first general weight function was probably given by Rice (1985) who evaluated the tensile mode weight function for a half-plane crack subjected to an arbitrarily located force. However, the result was given in integral form when the force direction was parallel to the crack plane and analytically when the direction was perpendicular to the plane. Recently, the derivation of the general weight functions for the penny-shaped and the half-plane crack have been given by Bueckner (1987) for an isotropic body. The analysis was extended by Gao (1989) to the circular external crack. In both of these analyses, explicit expressions for the weight functions were given only when the forces were located on a crack face (the crack face weight functions).

In the present analysis, weight functions for the penny-shaped crack are again considered. Use is made of some recent results by Fabrikant (1989) who derived closed form expressions for the elastic field of a penny-shaped crack in a transversely isotropic body loaded by point forces on its faces. These solutions, coupled with the reciprocal theorem, are used to derive closed form expressions in terms of elementary functions for the crack opening displacement of a penny-shaped crack in a transversely isotropic body loaded by an arbitrarily located point force. Explicit closed form expressions are obtained for the general weight functions of a penny-shaped crack in a transversely isotropic body by a limiting procedure. Such explicit expressions have not been given previously, even for the isotropic case.

POINT FORCE LOADING APPLIED TO A CIRCULAR CRACK

Consider a transversely isotropic space weakened by an internal circular crack of radius a in the plane $z = 0$. Let the crack be subjected to the action of two equal normal concentrated forces P applied in opposite directions at the points $(\rho_0, \phi_0, 0^{\pm})$, $\rho_0 < a$ as shown in Fig. 1(a). A complete solution for the field of displacements in elementary functions for $z > 0$ is (Fabrikant, 1989)

$$u = \frac{2}{\pi} HP \left[\frac{\gamma_1}{m_1 - 1} f_1(z_1) + \frac{\gamma_2}{m_2 - 1} f_1(z_2) \right], \tag{1}$$

$$w = \frac{2}{\pi} HP \left[\frac{m_1}{m_1 - 1} f_2(z_1) + \frac{m_2}{m_2 - 1} f_2(z_2) \right], \tag{2}$$

where $u = u_x + iu_y$ and u_x, u_y, w are the displacements in the x, y, z directions respectively. Here γ_1, γ_2, γ_3, m_1, m_2 are parameters relating to the five transversely isotropic elastic constants A_{11}, A_{13}, A_{33}, A_{44} and A_{66} used in Fabrikant (1989). The constant γ_3 is real while γ_1 and γ_2 are complex conjugate, as are m_1 and m_2. The quantities z_k are defined as $z_k = z/\gamma_k$ for $k = 1, 2, 3$, and hence z_1 and z_2 are complex conjugate also whereas z_3 is real. The

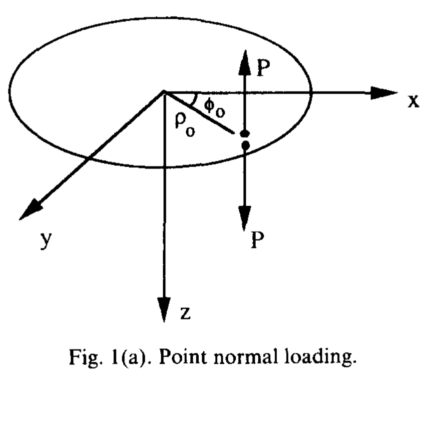

Fig. 1(a). Point normal loading.

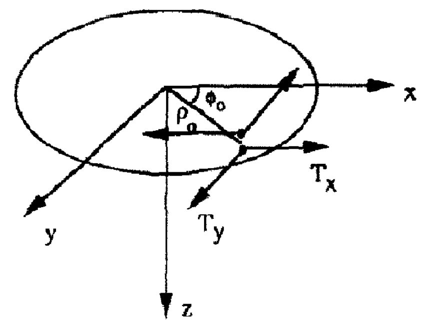

Fig. 1(b). Point shear loading.

elastic constant H is real and is defined in eqn (14). The interested reader is referred to Fabrikant (1988, 1989) or Hanson (1992) for further details.

If two equal tangential concentrated forces $T = T_x + iT_y$ are applied to the crack faces antisymmetrically at the points $(\rho_0, \phi_0, 0^\pm)$, $\rho_0 < a$, as shown in Fig. 1(b), a complete solution for the field of displacements in elementary functions for $z > 0$ is

$$u = \frac{H\gamma_1\gamma_2}{\pi}\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{-\left[f_2(z_k)+\frac{G_2}{G_1}\bar{f}_7(z_k)\right]T+\left[f_{16}(z_k)+\frac{G_2}{G_1}f_8(z_k)\right]\bar{T}\right\}$$
$$+\frac{\beta}{\pi}\left\{\left[f_2(z_3)-\frac{G_2}{G_1}\bar{f}_7(z_3)\right]T+\left[f_{16}(z_3)-\frac{G_2}{G_1}f_8(z_3)\right]\bar{T}\right\}, \quad (3)$$

$$w = \frac{2}{\pi}H\gamma_1\gamma_2\,\mathrm{Re}\left\{\sum_{k=1}^{2}\frac{m_k}{(m_k-1)\gamma_k}\left[\bar{f}_1(z_k)+\frac{G_2}{G_1}\bar{f}_9(z_k)\right]T\right\}. \quad (4)$$

Here Re indicates the real part and the functions $f_i(z)$ are given as

$$f_1(z) = \frac{1}{\bar{q}}\left[\frac{(a^2-\rho_0^2)^{1/2}}{\bar{s}}\tan^{-1}\left(\frac{\bar{s}}{(l_2^2-a^2)^{1/2}}\right)-\frac{z}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right)\right], \quad (5)$$

$$f_2(z) = \frac{1}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right), \quad (6)$$

$$f_7(z) = \frac{ha^2}{s^2}\left[\frac{3}{s^2}-\frac{t}{l_2^2-a^2t}-\frac{3(l_2^2-a^2)^{1/2}}{s^3}\tan^{-1}\left(\frac{s}{(l_2^2-a^2)^{1/2}}\right)\right], \quad (7)$$

$$f_8(z) = \frac{1}{\bar{q}}(a^2-\rho_0^2)^{1/2}\left\{\frac{(\bar{\zeta}-1)^{1/2}}{\bar{q}}\left[\tan^{-1}\left(\frac{1}{(\bar{\zeta}-1)^{1/2}}\right)-\tan^{-1}\left(\frac{(a^2-l_1^2)^{1/2}}{a(\bar{\zeta}-1)^{1/2}}\right)\right]\right.$$
$$\left.-\frac{e^{i\phi}}{\rho}\left[\frac{(a^2-l_1^2)^{1/2}}{a}\left(1+\frac{\rho^2}{l_2^2-\rho\rho_0 e^{i(\phi-\phi_0)}}\right)-1\right]\right\}, \quad (8)$$

$$f_9(z) = -\rho e^{i\phi}\frac{(a^2-\rho_0^2)^{1/2}}{a^3}\left\{\frac{1}{t}\sin^{-1}\left(\frac{a}{l_2}\right)+\frac{a(l_2^2-a^2)^{1/2}}{(1-t)(l_2^2-\rho\rho_0 e^{i(\phi-\phi_0)})}\right.$$
$$\left.-\frac{1}{t(1-t)^{3/2}}\tan^{-1}\left(\frac{a(1-t)^{1/2}}{(l_2^2-a^2)^{1/2}}\right)\right\}, \quad (9)$$

$$f_{16}(z) = \frac{1}{\bar{q}}\left\{\frac{R_0^2+z^2}{R_0\bar{q}}\tan^{-1}\left(\frac{h}{R_0}\right)+(a^2-\rho_0^2)^{1/2}\left[\frac{z}{\bar{s}}\left(\frac{\rho_0 e^{i\phi_0}}{\bar{s}^2}-\frac{2}{\bar{q}}\right)\tan^{-1}\left(\frac{\bar{s}}{(l_2^2-a^2)^{1/2}}\right)\right.\right.$$
$$\left.\left.+\frac{(\bar{\zeta}-1)^{1/2}}{\bar{q}}\left(\tan^{-1}\left(\frac{1}{(\bar{\zeta}-1)^{1/2}}\right)-\tan^{-1}\left(\frac{(a^2-l_1^2)^{1/2}}{a(\bar{\zeta}-1)^{1/2}}\right)\right)+\frac{e^{i\phi}}{\rho}\right]-\frac{e^{i\phi}ha^2}{\rho\bar{s}^2}\right\}. \quad (10)$$

Here the following notation was used:

$$R_0 = [\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2]^{1/2}, \quad q = \rho e^{i\phi}-\rho_0 e^{i\phi}, \quad (11)$$

$$l_1(a,\rho,z) \equiv l_1(a) = \tfrac{1}{2}\{[(\rho+a)^2+z^2]^{1/2}-[(\rho-a)^2+z^2]^{1/2}\},$$
$$l_2(a,\rho,z) \equiv l_2(a) = \tfrac{1}{2}\{[(\rho+a)^2+z^2]^{1/2}+[(\rho-a)^2+z^2]^{1/2}\}, \quad (12)$$

$$h = \sqrt{a^2-l_1^2}\sqrt{a^2-\rho_0^2}/a, \quad s = \sqrt{a^2-\rho\rho_0 e^{i(\phi-\phi_0)}},$$
$$t = \frac{\rho\rho_0}{a^2}e^{i(\phi-\phi_0)}, \quad \zeta = \frac{\rho}{\rho_0}e^{i(\phi-\phi_0)}. \quad (13)$$

An overbar indicates the complex conjugate value and the transversely isotropic elastic constants in (1–4) are defined as

$$G_1 = \beta+\gamma_1\gamma_2 H, \quad G_2 = \beta-\gamma_1\gamma_2 H,$$
$$H = \frac{(\gamma_1+\gamma_2)A_{11}}{2\pi(A_{11}A_{33}-A_{13}^2)}, \quad \beta = \frac{\gamma_3}{2\pi A_{44}}. \quad (14)$$

OPENING MODE DISPLACEMENTS AND STRESS INTENSITY FACTORS FOR AN ARBITRARY FORCE

Consider two systems in equilibrium: let a concentrated force Q_z be applied at an arbitrary point (ρ,ϕ,z) in the positive z direction, while in the second system two equal concentrated opening forces P are applied normal to the crack faces in opposite directions at the points $(\rho_0,\phi_0,0^\pm)$. Denote the normal displacement in the space due to the forces P by w_P, while w_{Q_z} is the crack opening displacement due to force Q_z. Note that the term "crack opening displacement" is used here to denote the difference between the normal displacements of the crack faces. Application of the reciprocal theorem to the two systems yields

$$Pw_{Q_z} = Q_z w_P, \quad (15)$$

which gives the crack opening displacement

$$w_{Q_z}(\rho_0,\phi_0) = \frac{2}{\pi}HQ_z\left[\frac{m_1}{m_1-1}f_2(z_1)+\frac{m_2}{m_2-1}f_2(z_2)\right], \quad (16)$$

with f_2 defined by (6).

Similarly we can consider two other systems in equilibrium. The displacement w_Q is produced by a concentrated force $Q = Q_x+iQ_y$ applied at an arbitrary point (ρ,ϕ,z) and directed perpendicular to the z axis in the positive x and y directions, while the tangential displacement u_P is due to the opening forces P applied normal to the crack faces in opposite directions at the points $(\rho_0,\phi_0,0^\pm)$. The application of the reciprocal theorem for Q_x and Q_y separately will give the following crack opening displacement:

$$w_{Q_z}(\rho_0, \phi_0) = \frac{2}{\pi} HQ_x \mathrm{Re}\left[\frac{\gamma_1}{m_1 - 1} f_1(z_1) + \frac{\gamma_2}{m_2 - 1} f_1(z_2)\right], \tag{17}$$

$$w_{Q_y}(\rho_0, \phi_0) = \frac{2}{\pi} HQ_y \mathrm{Im}\left[\frac{\gamma_1}{m_1 - 1} f_1(z_1) + \frac{\gamma_2}{m_2 - 1} f_1(z_2)\right], \tag{18}$$

with f_1 defined by (5).

The SIF can be determined by (Fabrikant, 1989)

$$K_1(\phi_0) = \frac{1}{8\pi H} \lim_{\rho_0 \to a} \frac{w_Q(\rho_0, \phi_0)}{(a - \rho_0)^{1/2}}, \tag{19}$$

and we obtain K_1 due to Q_z, Q_x and Q_y.

$$K_1 = \frac{Q_z}{2\pi^2 (2a)^{1/2}} \sum_{k=1}^{2} \frac{m_k}{m_k - 1} f_2^*(z_k), \tag{20}$$

$$K_1 = \frac{Q_x}{2\pi^2 (2a)^{1/2}} \mathrm{Re}\left[\sum_{k=1}^{2} \frac{\gamma_k}{m_k - 1} f_1^*(z_k)\right], \tag{21}$$

$$K_1 = \frac{Q_y}{2\pi^2 (2a)^{1/2}} \mathrm{Im}\left[\sum_{k=1}^{2} \frac{\gamma_k}{m_k - 1} f_1^*(z_k)\right]. \tag{22}$$

Here the functions $f_i^*(z)$ are given as

$$f_1^*(z_k) = \frac{1}{\bar{q}}\left[\frac{a}{\bar{s}} \tan^{-1}\left(\frac{\bar{s}}{(l_{2k}^2 - a^2)^{1/2}}\right) - \frac{z_k (a^2 - l_{1k}^2)^{1/2}}{R_k^2}\right], \tag{23}$$

$$f_2^*(z_k) = \frac{(a^2 - l_{1k}^2)^{1/2}}{R_k^2}, \tag{24}$$

where

$$R_k = [\rho^2 + a^2 - 2\rho a \cos(\phi - \phi_0) + z_k^2]^{1/2},$$

$$q = \rho\, \mathrm{e}^{i\phi} - a\, \mathrm{e}^{i\phi_0}, \quad s = \sqrt{a^2 - a\rho\, \mathrm{e}^{i(\phi - \phi_0)}}, \tag{25}$$

$$l_{1k} = \tfrac{1}{2}\{[(\rho + a)^2 + z_k^2]^{1/2} - [(\rho - a)^2 + z_k^2]^{1/2}\},$$

$$l_{2k} = \tfrac{1}{2}\{[(\rho + a)^2 + z_k^2]^{1/2} + [(\rho - a)^2 + z_k^2]^{1/2}\}. \tag{26}$$

The explanation for the evaluation of the required limits is given in the Appendix.

COMBINED SECOND AND THIRD MODE STRESS INTENSITY FACTORS

The solution for this case can still be obtained in an elementary way by using the reciprocal theorem as above. However, prior to it we shall show how the reciprocal theorem can be used in the case of complex forces and displacements.

Consider two systems in equilibrium. The first one is an elastic space weakened by an internal circular crack, with two equal and oppositely directed tangential forces $T = T_x + iT_y$ applied at the points $(\rho_0, \phi_0, 0^{\pm})$ on the crack faces [Fig. 1(b)]. The second system is the same space, with the crack faces tractions free, and the horizontal force $Q = Q_x + iQ_y$

applied at the point (ρ, ϕ, z) in the positive x and y directions. For simplicity of the transformation to follow, we present eqn (3) in a generalized form

$$u = A_1 T + A_2 \bar{T}. \tag{27}$$

Here A_1 and A_2 are the coefficients of T and $\bar{T}$ respectively. The tangential displacements at the point (ρ, ϕ, z) in the x and y directions caused by T_x will be

$$(u_x)_{T_x} = T_x \mathrm{Re}(A_1 + A_2), \quad (u_y)_{T_x} = T_x \,\mathrm{Im}(A_1 + A_2). \tag{28}$$

Similarly the displacements due to the forces T_y are

$$(u_x)_{T_y} = T_y \mathrm{Re}[(A_1 - A_2)i] = -T_y \mathrm{Im}(A_1 - A_2),$$

$$(u_y)_{T_y} = T_y \mathrm{Im}[(A_1 - A_2)i] = T_y \mathrm{Re}(A_1 - A_2). \tag{29}$$

We denote the tangential displacement discontinuity in the x direction due to force Q_x as Δ_x. According to the reciprocal theorem, we have

$$(\Delta_x)_{Q_x} = Q_x \mathrm{Re}(A_1 + A_2). \tag{30}$$

The remaining three equations are obtained in a similar manner, and they are

$$(\Delta_y)_{Q_x} = -Q_x \mathrm{Im}(A_1 - A), \quad (\Delta_x)_{Q_y} = Q_y \mathrm{Im}(A_1 + A_2), \quad (\Delta_y)_{Q_y} = Q_y \mathrm{Re}(A_1 - A_2). \tag{31}$$

Adding equation (30) with the first expression of (31) multiplied by i yields

$$(\Delta)_{Q_x} = (\Delta_x)_{Q_x} + i(\Delta_y)_{Q_x} = Q_x[\bar{A}_1 + A_2]. \tag{32}$$

A similar operation with the second and the third expressions in equations (31) results in

$$(\Delta)_{Q_y} = (\Delta_x)_{Q_y} + i(\Delta_y)_{Q_y} = Q_y[i\bar{A}_1 - iA_2]. \tag{33}$$

And finally, summation of equations (32) and (33) results in

$$\Delta_Q = (\Delta)_{Q_x} + (\Delta)_{Q_y} = \bar{A}_1 Q + A_2 \bar{Q}. \tag{34}$$

A comparison of eqns (27) and (34) suggests that we can obtain the tangential displacement discontinuity at the point $(\rho_0, \phi_0, 0)$ due to a tangential force Q applied at the point (ρ, ϕ, z) by using the expression for tangential displacements at the point (ρ, ϕ, z) due to a pair of equal and oppositely directed tangential forces T applied at the points $(\rho_0, \phi_0, 0^{\pm})$. The procedure is to substitute Q instead of T, and replace the coefficient of Q by its complex conjugate. Using this rule, we have from eqn (3)

$$\Delta_Q = \frac{H\gamma_1\gamma_2}{\pi} \sum_{k=1}^{2} \frac{1}{\bar{m}_k - 1} \left\{ -\left[\bar{f}_2(z_k) + \frac{G_2}{G_1} f_7(z_k) \right] Q + \left[f_{16}(z_k) + \frac{G_2}{G_1} f_8(z_k) \right] \bar{Q} \right\}$$
$$+ \frac{\beta}{\pi} \left\{ \left[f_2(z_3) - \frac{G_2}{G_1} f_7(z_3) \right] Q + \left[f_{16}(z_3) - \frac{G_2}{G_1} f_8(z_3) \right] \bar{Q} \right\}. \tag{35}$$

Now consider two other systems in equilibrium. The first system is an elastic space weakened by an internal circular crack, with two equal and oppositely directed tangential forces $T = T_x + iT_y$ applied at the points $(\rho_0, \phi_0, 0^{\pm})$ on the crack faces [Fig. 1(b)]. The second one is the same space, with the crack faces traction free, and the vertical force Q_z

applied at the point (ρ, ϕ, z). Let us, for the transformation to follow, present equation (4) in a generalized form

$$w = \mathrm{Re}\,(BT) = \tfrac{1}{2}(BT + \bar{B}\bar{T}). \tag{36}$$

Here B is the coefficient of T. The normal displacement at the point (ρ, ϕ, z) in the z direction due to antisymmetric forces T_x will be

$$w_{T_x} = T_x \tfrac{1}{2}(B + \bar{B}) = T_x \,\mathrm{Re}\,(B). \tag{37}$$

The respective displacement due to antisymmetric forces T_y is

$$w_{T_y} = iT_y \tfrac{1}{2}(B - \bar{B}) = -T_y \,\mathrm{Im}\,(B). \tag{38}$$

If we denote the tangential displacement discontinuity in the x and y directions due to a force Q_z as Δ_x and Δ_y, respectively, then according to the reciprocal theorem, we have

$$(\Delta_x)_{Q_z} = Q_z \,\mathrm{Re}\,(B), \quad (\Delta_y)_{Q_z} = -Q_z \,\mathrm{Im}\,(B). \tag{39}$$

Summation of the first expression of equation (39) with the second one multiplied by i results in

$$\Delta_{Q_z} = (\Delta_x)_{Q_z} + i(\Delta_y)_{Q_z} = Q_z[\mathrm{Re}\,(B) - i\,\mathrm{Im}\,(B)] = \bar{B}Q_z. \tag{40}$$

A comparison of equations (36) and (40) suggests that we can obtain the tangential displacement discontinuity at the point $(\rho_0, \phi_0, 0)$ due to a normal force Q_z applied at the point (ρ, ϕ, z) by using the expression for tangential displacements at the point (ρ, ϕ, z) caused by a pair of equal and oppositely directed tangential forces T applied at the points $(\rho_0, \phi_0, 0^{\pm})$. Hence we have from equation (4)

$$\Delta_{Q_z} = \frac{2}{\pi} Q_z H \gamma_1 \gamma_2 \sum_{k=1}^{2} \frac{\bar{m}_k}{(\bar{m}_k - 1)\bar{\gamma}_k} \left[f_1(z_k) + \frac{G_2}{G_1} f_9(z_k) \right]. \tag{41}$$

The stress intensity factors of the second and third kind can be expressed through the tangential displacement discontinuity (Fabrikant, 1989) as

$$K_2 + iK_3 = -\frac{a}{2\pi(G_1^2 - G_2^2)\sqrt{2a}} \lim_{\rho_0 \to a} \left[\frac{G_1 \,\mathrm{e}^{-i\phi_0}\Delta + G_2 \,\mathrm{e}^{i\phi_0}\bar{\Delta}}{(a^2 - \rho_0^2)^{1/2}} \right]. \tag{42}$$

Substitution of equations (35) and (41) in equation (42) and taking the limit will give the desired results for the stress intensity factors.

For application of Q_x

$$\begin{aligned} K_2 = -\frac{Q_x a}{4\pi^2\sqrt{2a}} \mathrm{Re} \Bigg\{ & \frac{G_1 \,\mathrm{e}^{-i\phi_0}}{G_1 + G_2} \sum_{k=1}^{2} \frac{1}{\bar{m}_k - 1} (-f_3^*(z_k) + f_4^*(z_k)) \\ & + \frac{G_2 \,\mathrm{e}^{i\phi_0}}{G_1 + G_2} \sum_{k=1}^{2} \frac{1}{m_k - 1} (-\bar{f}_3^*(z_k) + \bar{f}_4^*(z_k)) + \frac{G_1 \,\mathrm{e}^{-i\phi_0}}{G_1 - G_2} (f_5^*(z_3) + f_6^*(z_3)) \\ & + \frac{G_2 \,\mathrm{e}^{i\phi_0}}{G_1 - G_2} (\bar{f}_5^*(z_3) + \bar{f}_6^*(z_3)) \Bigg\}, \end{aligned} \tag{43}$$

$$K_3 = -\frac{Q_x a}{4\pi^2\sqrt{2a}}\,\mathrm{Im}\left\{\frac{G_1\,\mathrm{e}^{-i\phi_0}}{G_1+G_2}\sum_{k=1}^{2}\frac{1}{\bar{m}_k-1}(-f_3^*(z_k)+f_4^*(z_k))\right.$$
$$+\frac{G_2\,\mathrm{e}^{i\phi_0}}{G_1+G_2}\sum_{k=1}^{2}\frac{1}{m_k-1}(-\bar{f}_3^*(z_k)+\bar{f}_4^*(z_k))+\frac{G_1\,\mathrm{e}^{-i\phi_0}}{G_1-G_2}(f_5^*(z_3)+f_6^*(z_3))$$
$$\left.+\frac{G_2\,\mathrm{e}^{i\phi_0}}{G_1-G_2}(\bar{f}_5^*(z_3)+\bar{f}_6^*(z_3))\right\}. \quad (44)$$

For application of Q_y

$$K_2 = -\frac{Q_y a}{4\pi^2\sqrt{2a}}\,\mathrm{Re}\left\{\frac{G_1 i\,\mathrm{e}^{-i\phi_0}}{G_1+G_2}\sum_{k=1}^{2}\frac{1}{\bar{m}_k-1}(-f_3^*(z_k)-f_4^*(z_k))\right.$$
$$+\frac{G_2 i\,\mathrm{e}^{i\phi_0}}{G_1+G_2}\sum_{k=1}^{2}\frac{1}{m_k-1}(\bar{f}_3^*(z_k)+\bar{f}_4^*(z_k))+\frac{G_1 i\,\mathrm{e}^{-i\phi_0}}{G_1-G_2}(f_5^*(z_3)-f_6^*(z_3))$$
$$\left.+\frac{G_2 i\,\mathrm{e}^{i\phi_0}}{G_1-G_2}(-\bar{f}_5^*(z_3)+\bar{f}_6^*(z_3))\right\}, \quad (45)$$

$$K_3 = -\frac{Q_y a}{4\pi^2\sqrt{2a}}\,\mathrm{Im}\left\{\frac{G_1 i\,\mathrm{e}^{-i\phi_0}}{G_1+G_2}\sum_{k=1}^{2}\frac{1}{\bar{m}_k-1}(-f_3^*(z_k)-f_4^*(z_k))\right.$$
$$+\frac{G_2 i\,\mathrm{e}^{i\phi_0}}{G_1+G_2}\sum_{k=1}^{2}\frac{1}{m_k-1}(\bar{f}_3^*(z_k)+\bar{f}_4^*(z_k))+\frac{G_1 i\,\mathrm{e}^{-i\phi_0}}{G_1-G_2}(f_5^*(z_3)-f_6^*(z_3))$$
$$\left.+\frac{G_2 i\,\mathrm{e}^{i\phi_0}}{G_1-G_2}(-\bar{f}_5^*(z_3)+\bar{f}_6^*(z_3))\right\}. \quad (46)$$

For application of Q_z

$$K_2 = -\frac{Q_z a}{4\pi^2\beta\sqrt{2a}}\,\mathrm{Re}\left\{\sum_{k=1}^{2}\frac{\bar{m}_k}{(\bar{m}_k-1)\bar{\gamma}_k}(G_1\bar{f}_7^*(z_k)+G_2 f_8^*(z_k))\right.$$
$$\left.+\sum_{k=1}^{2}\frac{m_k}{(m_k-1)\gamma_k}\left(G_2 f_7^*(z_k)+\frac{G_2^2}{G_1}\bar{f}_8^*(z_k)\right)\right\}, \quad (47)$$

$$K_2 = -\frac{Q_z a}{4\pi^2\beta\sqrt{2a}}\,\mathrm{Im}\left\{\sum_{k=1}^{2}\frac{\bar{m}_k}{(\bar{m}_k-1)\bar{\gamma}_k}(G_1\bar{f}_7^*(z_k)+G_2 f_8^*(z_k))\right.$$
$$\left.+\sum_{k=1}^{2}\frac{m_k}{(m_k-1)\gamma_k}\left(G_2 f_7^*(z_k)+\frac{G_2^2}{G_1}\bar{f}_8^*(z_k)\right)\right\}. \quad (48)$$

Here the functions $f_i^*(z)$ are given as

$$f_3^*(z_k) = \frac{(a^2-l_{1k}^2)^{1/2}}{R_k^2 a}+\frac{G_2}{G_1}\frac{a(a^2-l_{1k}^2)^{1/2}}{s^2}\left[\frac{3}{s^2}-\frac{\rho\,\mathrm{e}^{i(\phi-\phi_0)}}{a(l_{2k}^2-a\rho\,\mathrm{e}^{i(\phi-\phi_0)})}\right.$$
$$\left.-\frac{3(l_{2k}^2-a^2)^{1/2}}{s^3}\tan^{-1}\left(\frac{s}{(l_{2k}^2-a^2)^{1/2}}\right)\right], \quad (49)$$

$$f_4^*(z_k) = \frac{1}{\bar{q}}\left\{\frac{R_k^2+z_k^2}{R_k^2\bar{q}}\frac{(a^2-l_{1k}^2)^{1/2}}{a}-\frac{3z_k}{\bar{s}\bar{q}}\tan^{-1}\left(\frac{\bar{s}}{(l_{2k}^2-a^2)^{1/2}}\right)\right.$$

$$-\frac{(\bar{\zeta}-1)^{1/2}}{\bar{q}}\tan^{-1}\left(\frac{(a^2-l_{1k}^2)^{1/2}}{a(\bar{\zeta}-1)^{1/2}}\right)-\frac{e^{i\phi}}{\rho}\left(\frac{a(a^2-l_{1k}^2)^{1/2}}{\bar{s}^2}-1\right)$$

$$-\frac{G_2}{G_1}\left[\frac{(\bar{\zeta}-1)^{1/2}}{\bar{q}}\tan^{-1}\left(\frac{(a^2-l_{1k}^2)^{1/2}}{a(\bar{\zeta}-1)^{1/2}}\right)+\frac{e^{i\phi}\rho(a^2-l_{1k}^2)^{1/2}}{a(l_{2k}^2-a\rho\, e^{i(\phi-\phi_0)})}\right.$$

$$\left.+\frac{e^{i\phi}}{\rho}\left(\frac{(a^2-l_{1k}^2)^{1/2}}{a}-1\right)\right]\Bigg\}, \tag{50}$$

$$f_5^*(z_3)=\frac{(a^2-l_{13}^2)^{1/2}}{R_3^2 a}-\frac{G_2}{G_1}\frac{a(a^2-l_{13}^2)^{1/2}}{s^2}\left[\frac{3}{s^2}-\frac{\rho\, e^{i(\phi-\phi_0)}}{a(l_{23}^2-a\rho\, e^{i(\phi-\phi_0)})}\right.$$

$$\left.-\frac{3(l_{23}^2-a^2)^{1/2}}{s^3}\tan^{-1}\left(\frac{s}{(l_{23}^2-a^2)^{1/2}}\right)\right], \tag{51}$$

$$f_6^*(z_3)=\frac{1}{\bar{q}}\left\{\frac{R_3^2+z_3^2}{R_3^2\bar{q}}\frac{(a^2-l_{13}^2)^{1/2}}{a}-\frac{3z_3}{\bar{s}\bar{q}}\tan^{-1}\left(\frac{\bar{s}}{(l_{23}^2-a^2)^{1/2}}\right)\right.$$

$$-\frac{(\bar{\zeta}-1)^{1/2}}{\bar{q}}\tan^{-1}\left(\frac{(a^2-l_{13}^2)^{1/2}}{a(\bar{\zeta}-1)^{1/2}}\right)-\frac{e^{i\phi}}{\rho}\left(\frac{a(a^2-l_{13}^2)^{1/2}}{\bar{s}^2}-1\right)$$

$$+\frac{G_2}{G_1}\left[\frac{(\bar{\zeta}-1)^{1/2}}{\bar{q}}\tan^{-1}\left(\frac{(a^2-l_{13}^2)^{1/2}}{a(\bar{\zeta}-1)^{1/2}}\right)+\frac{e^{i\phi}\rho(a^2-l_{13}^2)^{1/2}}{a(l_{23}^2-a\rho\, e^{i(\phi-\phi_0)})}\right.$$

$$\left.\left.+\frac{e^{i\phi}}{\rho}\left(\frac{(a^2-l_{13}^2)^{1/2}}{a}-1\right)\right]\right\}, \tag{52}$$

$$f_7^*(z_k)=\frac{z_k(a^2-l_{1k}^2)^{1/2}}{R_k^2 s^2}-\frac{a}{s^3}\tan^{-1}\left(\frac{s}{(l_{2k}^2-a^2)^{1/2}}\right), \tag{53}$$

$$f_8^*(z_k)=\frac{a}{s^3}\tan^{-1}\left(\frac{s}{(l_{2k}^2-a^2)^{1/2}}\right)-\frac{1}{a^2}\sin^{-1}\left(\frac{a}{l_{2k}}\right)-\frac{e^{i(\phi-\phi_0)}\rho(l_{2k}^2-a^2)^{1/2}}{s^2(l_{2k}^2-\rho a\, e^{i(\phi-\phi_0)})}, \tag{54}$$

where R_k, q, s, l_{1k} and l_{2k} are defined as in eqns (25)–(26), while

$$R_3=[\rho^2+a^2-2\rho a\cos(\phi-\phi_0)+z_3^2]^{1/2},\quad \zeta=\frac{\rho}{a}e^{i(\phi-\phi_0)},$$

$$l_{13}=\tfrac{1}{2}\{[(\rho+a)^2+z_3^2]^{1/2}-[(\rho-a)^2+z_3^2]^{1/2}\},$$

$$l_{23}=\tfrac{1}{2}\{[(\rho+a)^2+z_3^2]^{1/2}+[(\rho-a)^2+z_3^2]^{1/2}\}. \tag{55}$$

Thus the expressions in eqns (20)–(22), (43)–(48) present all modes of stress intensity factors, the so called "weight functions" for a penny-shaped crack, in a transversely isotropic body.

ISOTROPIC SOLUTION

All of the results obtained before are valid for isotropic solids, provided that we take

$$\gamma_1=\gamma_2=\gamma_3=1,\quad H=\frac{1-\nu^2}{\pi E},\quad \beta=\frac{1+\nu}{\pi E},\quad G_1=\frac{(2-\nu)(1+\nu)}{\pi E},\quad G_2=\frac{\nu(1+\nu)}{\pi E}, \tag{56}$$

where E is the elastic modulus, and ν is Poisson's ratio. The limits were computed according to L'Hôpital's rule. The following scheme was used:

$$\lim_{\gamma_1\to\gamma_2\to 1}\sum_{k=1}^{2}\frac{m_k}{(m_k-1)}f(z_k)=f(z)-\frac{z}{2(1-\nu)}f'(z), \tag{57}$$

$$\lim_{\gamma_1\to\gamma_2\to 1}\sum_{k=1}^{2}\frac{\gamma_k}{m_k-1}f(z_k)=-\frac{(1-2\nu)f(z)+zf'(z)}{2(1-\nu)}, \tag{58}$$

$$\lim_{\gamma_1\to\gamma_2\to 1}\sum_{k=1}^{2}\frac{m_k}{(m_k-1)\gamma_k}f(z_k)=\frac{(1-2\nu)f(z)-zf'(z)}{2(1-\nu)}, \tag{59}$$

$$\lim_{\gamma_1\to\gamma_2\to 1}\sum_{k=1}^{2}\frac{1}{m_k-1}f(z_k)=-f(z)-\frac{z}{2(1-\nu)}f'(z). \tag{60}$$

Here the following relationships were used

$$\lim_{\gamma_1\to\gamma_2\to 1}m_1=1,\quad \lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{\partial m_1}{\partial\gamma_1}\right]=2(1-\nu), \tag{61}$$

and the symbol (′) indicates differentiation with respect to z. Application of (57)–(58) to the expressions (20)–(22) will give us K_1 SIF due to Q_z, Q_x and Q_y.

For Q_z

$$K_1=\frac{Q_z(a^2-l_1^2)^{1/2}}{2\pi^2\sqrt{2a}R^2}f_1^{**}(z). \tag{62}$$

For Q_x

$$K_1=-\frac{Q_x(a^2-l_1^2)^{1/2}}{4\pi^2\sqrt{2a}(1-\nu)}\operatorname{Re}[f_2^{**}(z)]. \tag{63}$$

For Q_y

$$K_1=-\frac{Q_y(a^2-l_1^2)^{1/2}}{4\pi^2\sqrt{2a}(1-\nu)}\operatorname{Im}[f_2^{**}(z)]. \tag{64}$$

Here the functions $f_i^{**}(z)$ are given as

$$f_1^{**}(z)=1+\frac{1}{1-\nu}\left(\frac{z^2}{R^2}-\frac{\rho^2-l_1^2}{2(l_2^2-l_1^2)}\right), \tag{65}$$

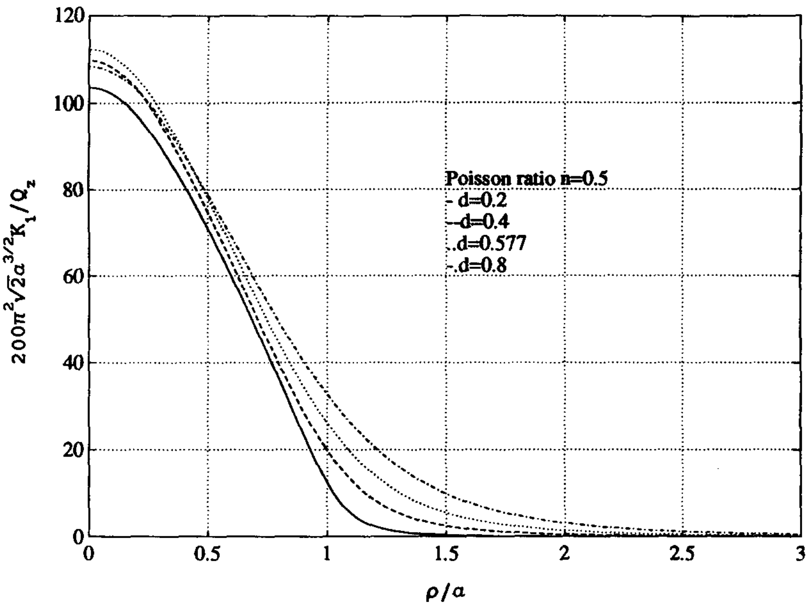

Fig. 2. Stress intensity factor K_1 due to force Q_z at an arbitrary point in space.

$$f_2^{**}(z) = \frac{1}{\bar{q}}\left[(1-2\nu)\left(\frac{a}{(a^2-l_1^2)^{1/2}\bar{s}}\tan^{-1}\left(\frac{\bar{s}}{(l_2^2-a^2)^{1/2}}\right)-\frac{z}{R^2}\right)\right.$$
$$\left.+\frac{z}{R^2}\left(\frac{2z^2}{R^2}-\frac{\rho^2-l_1^2}{l_2^2-l_1^2}-1\right)-\frac{zl_2^2}{(l_2^2-a^2+\bar{s}^2)(l_2^2-l_1^2)}\right]. \quad (66)$$

Figures 2 and 3 give graphical representations of K_1 due to Q_z and Q_x forces, respectively.

If we apply formulae (59)–(60) to the expressions in (43)–(48) we will obtain K_2 and K_3 due to Q_z, Q_x and Q_y.

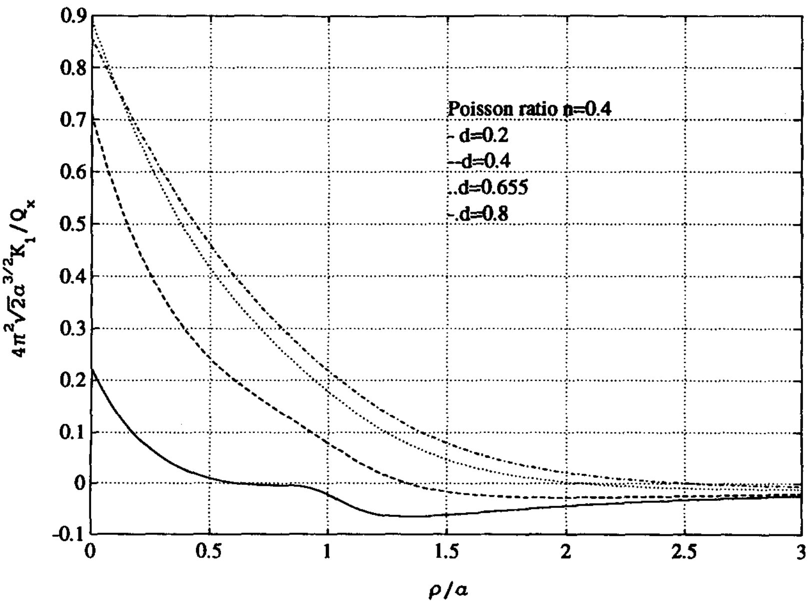

Fig. 3. Stress intensity factor K_1 due to force Q_x at an arbitrary point in space.

For Q_z

$$K_2 = -\frac{Q_z(a^2-l_1^2)^{1/2}}{8\pi^2(1-\nu)\sqrt{2a}}\,\mathrm{Re}\left[(2-\nu)\bar{f}_3^{**}(z)+\nu f_3^{**}(z)+\nu f_4^{**}(z)+\frac{\nu^2}{2-\nu}\bar{f}_4^{**}(z)\right], \quad (67)$$

$$K_3 = -\frac{Q_z(a^2-l_1^2)^{1/2}}{8\pi^2(1-\nu)\sqrt{2a}}\,\mathrm{Im}\left[(2-\nu)\bar{f}_3^{**}(z)+\nu f_3^{**}(z)+\nu f_4^{**}(z)+\frac{\nu^2}{2-\nu}\bar{f}_4^{**}(z)\right]. \quad (68)$$

For Q_x

$$K_2 = -\frac{Q_x(a^2-l_1^2)^{1/2}}{8\pi^2(1-\nu)\sqrt{2a}}\,\mathrm{Re}\left\{e^{i\phi_0}\left[\nu(\bar{f}_5^{**}(z)-\bar{f}_6^{**}(z))+\frac{\nu}{2}(\bar{f}_7^{**}(z)-\bar{f}_8^{**}(z))\right]\right.$$
$$\left.+e^{-i\phi_0}\left[(2-\nu)(f_5^{**}(z)-f_6^{**}(z))+\frac{2-\nu}{2}(f_7^{**}(z)-f_8^{**}(z))\right]\right\}, \quad (69)$$

$$K_3 = -\frac{Q_x(a^2-l_1^2)^{1/2}}{8\pi^2(1-\nu)\sqrt{2a}}\,\mathrm{Im}\left\{e^{i\phi_0}\left[\nu(\bar{f}_5^{**}(z)-\bar{f}_6^{**}(z))+\frac{\nu}{2}(\bar{f}_7^{**}(z)-\bar{f}_8^{**}(z))\right]\right.$$
$$\left.+e^{-i\phi_0}\left[(2-\nu)(f_5^{**}(z)-f_6^{**}(z))+\frac{2-\nu}{2}(f_7^{**}(z)-f_8^{**}(z))\right]\right\}. \quad (70)$$

For Q_y

$$K_2 = -\frac{Q_y(a^2-l_1^2)^{1/2}}{8\pi^2(1-\nu)\sqrt{2a}}\,\mathrm{Re}\left\{-ie^{i\phi_0}\left[\nu(\bar{f}_5^{**}(z)+\bar{f}_6^{**}(z))+\frac{\nu}{2}(\bar{f}_7^{**}(z)+\bar{f}_8^{**}(z))\right]\right.$$
$$\left.+ie^{-i\phi_0}\left[(2-\nu)(f_5^{**}(z)+f_6^{**}(z))+\frac{2-\nu}{2}(f_7^{**}(z)+f_8^{**}(z))\right]\right\}, \quad (71)$$

$$K_3 = -\frac{Q_y(a^2-l_1^2)^{1/2}}{8\pi^2(1-\nu)\sqrt{2a}}\,\mathrm{Im}\left\{-ie^{i\phi_0}\left[\nu(\bar{f}_5^{**}(z)+\bar{f}_6^{**}(z))+\frac{\nu}{2}(\bar{f}_7^{**}(z)+\bar{f}_8^{**}(z))\right]\right.$$
$$\left.+ie^{-i\phi_0}\left[(2-\nu)(f_5^{**}(z)+f_6^{**}(z))+\frac{2-\nu}{2}(f_7^{**}(z)+f_8^{**}(z))\right]\right\}. \quad (72)$$

Here the functions $f_i^{**}(z)$ are given as

$$f_3^{**}(z) = \frac{a}{s^2}\left[(1-2\nu)\left(\frac{z}{R^2}-\frac{a}{s(a^2-l_1^2)^{1/2}}\tan^{-1}\left(\frac{s}{(l_2^2-a^2)^{1/2}}\right)\right)\right.$$
$$\left.-\frac{z}{R^2}\left(1+\frac{\rho^2-l_1^2}{l_2^2-l_1^2}-\frac{2z^2}{R^2}\right)-\frac{zl_2^2}{(l_2^2-\rho a\,e^{i(\phi-\phi_0)})(l_2^2-l_1^2)}\right], \quad (73)$$

$$f_4^{**}(z) = (1-2\nu)\left\{\frac{1}{a(a^2-l_1^2)^{1/2}}\left[\frac{a^3}{s^3}\tan^{-1}\left(\frac{s}{(l_2^2-a^2)^{1/2}}\right)-\sin^{-1}\left(\frac{a}{l_2}\right)\right]\right.$$
$$\left.-\frac{z\rho a^2\,e^{i(\phi-\phi_0)}}{(a^2-l_1^2)(l_2^2-\rho a\,e^{i(\phi-\phi_0)})s^2}\right\}+\frac{z\rho\,e^{i(\phi-\phi_0)}}{(l_2^2-\rho a\,e^{i(\phi-\phi_0)})(l_2^2-l_1^2)}$$
$$+\frac{2z\rho l_2^2\,e^{i(\phi-\phi_0)}}{(l_2^2-\rho a\,e^{i(\phi-\phi_0)})^2(l_2^2-l_1^2)}, \quad (74)$$

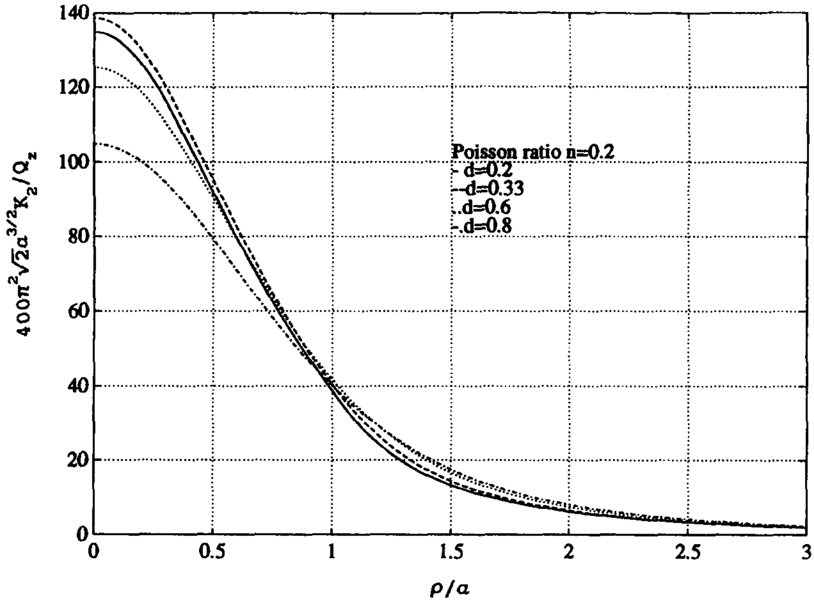

Fig. 4. Stress intensity factor K_2 due to force Q_z at an arbitrary point in space.

$$f_5^{**}(z) = (2-\nu)\frac{1}{R^2} - \frac{\nu^2}{2-\nu}\frac{a^2}{s^2}\left[\frac{3}{s^2} - \frac{\zeta}{l_2^2 - a\rho\,e^{i(\phi-\phi_0)}} - \frac{3(l_2^2-a^2)^{1/2}}{s^3}\tan^{-1}\left(\frac{s}{(l_2^2-a^2)^{1/2}}\right)\right], \quad (75)$$

$$f_6^{**}(z) = \nu\frac{1}{\bar{q}^2}\left[\frac{3(l_2^2-a^2)^{1/2}}{\bar{s}}\tan^{-1}\left(\frac{\bar{s}}{(l_2^2-a^2)^{1/2}}\right) - \frac{2R^2+z^2}{R^2} - \frac{\bar{q}\rho\,e^{i\phi}}{l_2^2 - a\rho\,e^{i(\phi-\phi_0)}}\right], \quad (76)$$

$$f_7^{**}(z) = \frac{1}{R^2}\left(\frac{\rho^2-l_1^2}{l_2^2-l_1^2} - \frac{2z^2}{R^2}\right) + \frac{\nu}{2-\nu}\frac{1}{s^2}\left[\frac{\rho^2-l_1^2}{l_2^2-l_1^2}\left(\frac{3a^2}{s^2} - \frac{a\rho\,e^{i(\phi-\phi_0)}}{l_2^2-a\rho\,e^{i(\phi-\phi_0)}}\right)\right.$$
$$- \frac{3a^2(l_2^2-a^2)^{1/2}}{s^3}\tan^{-1}\left(\frac{s}{(l_2^2-a^2)^{1/2}}\right) + \frac{2z^2 l_2^2 a\rho\,e^{i(\phi-\phi_0)}}{(l_2^2-a\rho\,e^{i(\phi-\phi_0)})^2(l_2^2-l_1^2)}$$
$$\left. + \frac{3z^2 l_2^2 a^2}{s^2(l_2^2-a\rho\,e^{i(\phi-\phi_0)})(l_2^2-l_1^2)}\right], \quad (77)$$

$$f_8^{**}(z) = \frac{1}{\bar{q}^2}\left[\frac{2z^2(R^2-z^2)}{R^4} + \frac{R^2+z^2}{R^2}\frac{\rho^2-l_1^2}{l_2^2-l_1^2} - \frac{3(l_2^2-a^2)^{1/2}}{\bar{s}}\tan^{-1}\left(\frac{\bar{s}}{(l_2^2-a^2)^{1/2}}\right)\right.$$
$$+ \frac{3z^2 l_2^2}{(l_2^2-a\rho\,e^{-i(\phi-\phi_0)})(l_2^2-l_1^2)} - \frac{2}{2-\nu}\frac{\rho^2-l_1^2}{l_2^2-l_1^2}\left(\frac{\bar{s}^2}{l_1^2-a\rho\,e^{-i(\phi-\phi_0)}}\right.$$
$$\left. - \frac{a\,e^{i(\phi-\phi_0)}}{\rho}\right) - \frac{\nu}{2-\nu}\left(\frac{\rho^2-l_1^2}{l_2^2-l_1^2} + \frac{\bar{q}\rho\,e^{i\phi}(\rho^2-l_1^2)}{(l_2^2-a\rho\,e^{i(\phi-\phi_0)})(l_2^2-l_1^2)}\right.$$
$$\left.\left. - \frac{2z^2 l_2^2\bar{q}\rho\,e^{i\phi}}{(l_2^2-a\rho\,e^{i(\phi-\phi_0)})^2(l_2^2-l_1^2)}\right)\right]. \quad (78)$$

Figures 4 and 5 display K_2 and K_3 due to force Q_z, while Figs 6 and 7 show K_2 and K_3 resulting from force Q_x.

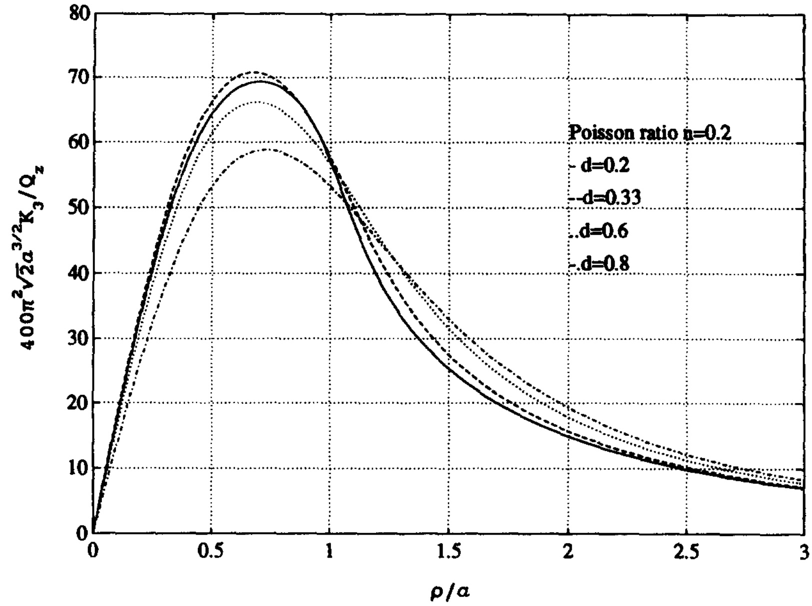

Fig. 5. Stress intensity factor K_3 due to force Q_z at an arbitrary point in space.

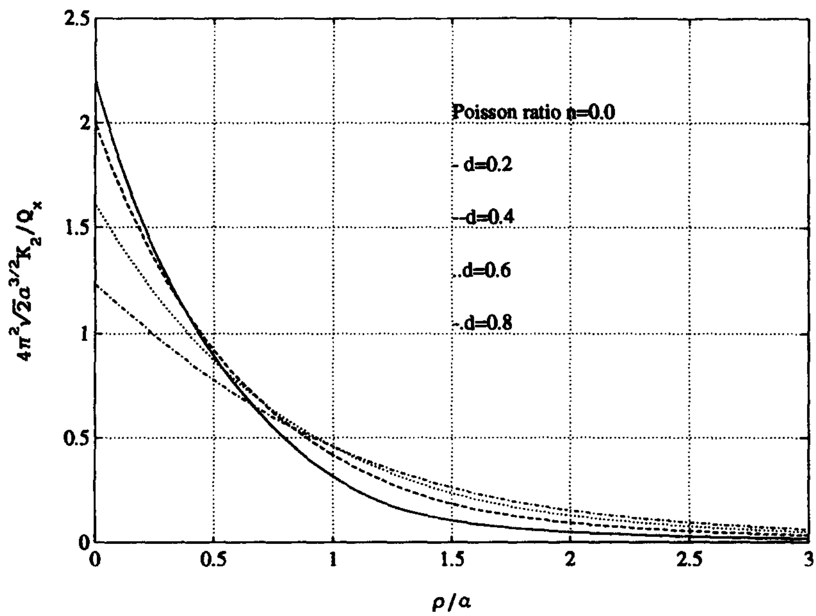

Fig. 6. Stress intensity factor K_2 due to force Q_x at an arbitrary point in space.

AXISYMMETRIC CASE AND COMPARISON WITH THE RESULTS REPORTED IN THE LITERATURE

It is quite interesting to consider the particular case of concentrated forces applied at the point on the vertical z axes when $\rho = 0$. In this case the expressions obtained in eqns (62)–(64) and (67)–(72) will drastically simplify and reduce to:

For Q_z

$$K_1 = \frac{Q_z(2a)^{1/2}}{4\pi^2(a^2+z^2)}\left[1+\frac{1}{1-\nu}\frac{z^2}{a^2+z^2}\right], \tag{79}$$

$$K_2 = -\frac{Q_z}{4\pi^2(1-\nu)(2a)^{1/2}}\left[(1-2\nu)\left(\frac{z}{a^2+z^2}-\frac{1}{a}\tan^{-1}\left(\frac{a}{z}\right)\right)-\frac{2za^2}{(a^2+z^2)^2}\right], \tag{80}$$

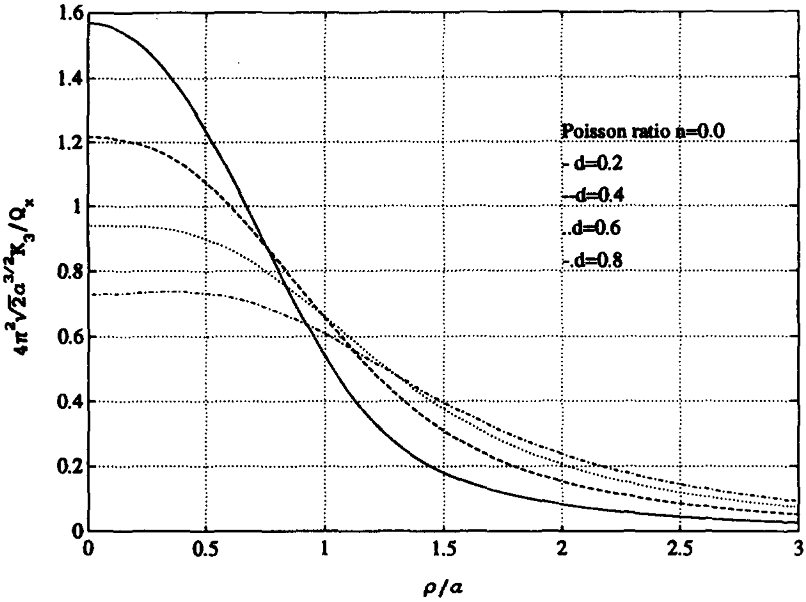

Fig. 7. Stress intensity factor K_3 due to force Q_x at an arbitrary point in space.

$$K_3 = 0. \tag{81}$$

For Q_x

$$K_1 = \frac{Q_x \cos\phi_0}{4\pi^2(2a)^{1/2}(1-\nu)}\left\{(1-2\nu)\left[\frac{1}{a}\tan^{-1}\left(\frac{a}{z}\right)-\frac{z}{a^2+z^2}\right]-\frac{2za^2}{(a^2+z^2)^2}\right\}, \tag{82}$$

$$K_2 = -\frac{Q_x\cos\phi_0}{2\pi^2(1-\nu)(2-\nu)(2a)^{3/2}}\left\{3(1-\nu)(1-2\nu)\left[\frac{z}{a}\tan^{-1}\left(\frac{a}{z}\right)-\frac{z^2}{a^2+z^2}\right]\right.$$
$$\left.+\frac{2a^2}{a^2+z^2}\left[2(1-\nu^2)-\frac{(2-\nu)z^2}{a^2+z^2}\right]\right\}. \tag{83}$$

$$K_3 = \frac{(1-2\nu)Q_x\sin\phi_0}{2\pi^2(2-\nu)(2a)^{3/2}}\left[3-\frac{3z}{a}\tan^{-1}\left(\frac{a}{z}\right)+\frac{a^2}{a^2+z^2}\right]. \tag{84}$$

For Q_y

$$K_1 = \frac{Q_y \sin\phi_0}{4\pi^2(2a)^{1/2}(1-\nu)}\left\{(1-2\nu)\left[\frac{1}{a}\tan^{-1}\left(\frac{a}{z}\right)-\frac{z}{a^2+z^2}\right]-\frac{2za^2}{(a^2+z^2)^2}\right\}, \tag{85}$$

$$K_2 = -\frac{Q_y\sin\phi_0}{2\pi^2(1-\nu)(2-\nu)(2a)^{3/2}}\left\{3(1-\nu)(1-2\nu)\left[\frac{z}{a}\tan^{-1}\left(\frac{a}{z}\right)-\frac{z^2}{a^2+z^2}\right]\right.$$
$$\left.+\frac{2a^2}{a^2+z^2}\left[2(1-\nu^2)-\frac{(2-\nu)z^2}{a^2+z^2}\right]\right\}, \tag{86}$$

$$K_3 = -\frac{(1-2\nu)Q_y\cos\phi_0}{2\pi^2(2-\nu)(2a)^{3/2}}\left[3-\frac{3z}{a}\tan^{-1}\left(\frac{a}{z}\right)+\frac{a^2}{a^2+z^2}\right]. \tag{87}$$

All of the results for this particular case are in perfect agreement with the known results

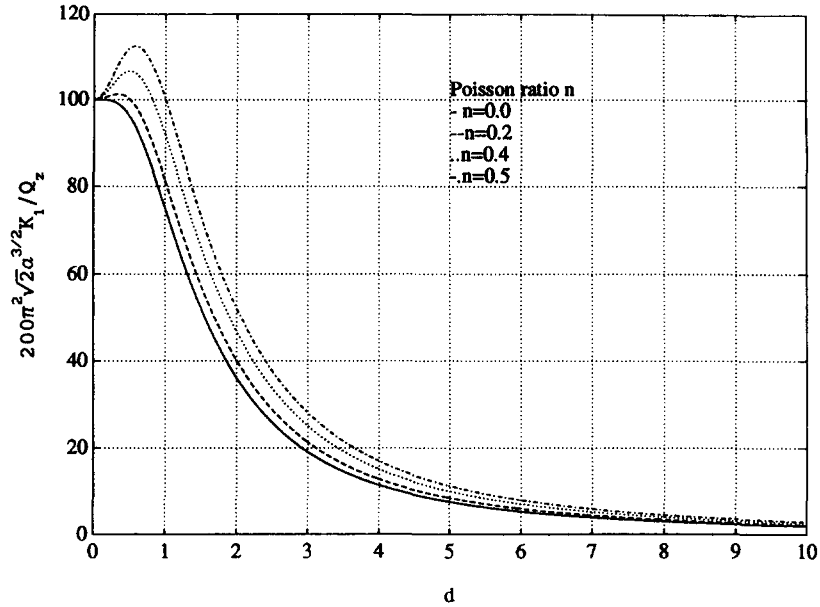

Fig. 8. Stress intensity factor K_1 due to force Q_z at a point on the normal axis.

given in Kassir and Sih (1975). (Because of SIF definitions our results differ by a factor of $\sqrt{2}$.) In spite of this, the graphical representations for eqns (79)–(84) given in Kassir and Sih (1975) are not quite correct and are repeated here as Figs 8–12.

For example, the calculation of the maximum of the function in equation (79) results in the following relationship

$$d = \sqrt{\frac{v}{2-v}}, \tag{88}$$

where $d = z/a$. As we can see from (88), when $v = 0.5$ the value of $d = 0.577$ corresponds to the maximum of K_1 SIF as shown in Fig. 8. It is interesting to note that in Fig. 2 at the point $\rho = 0$, for the same values of v and d, the maximum of K_1 is identical to that in Fig. 8.

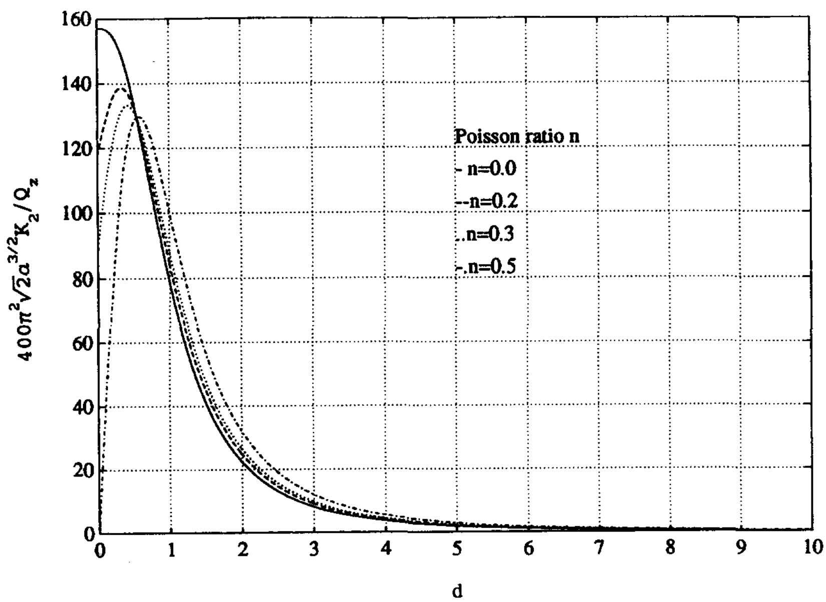

Fig. 9. Stress intensity factor K_2 due to force Q_z at a point on the normal axis.

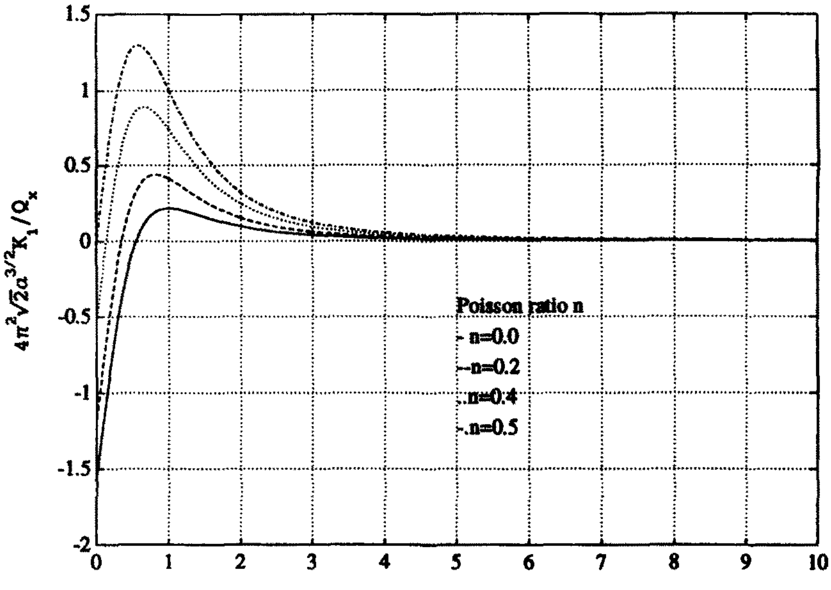

Fig. 10. Stress intensity factor K_1 due to force Q_x at a point on the normal axis.

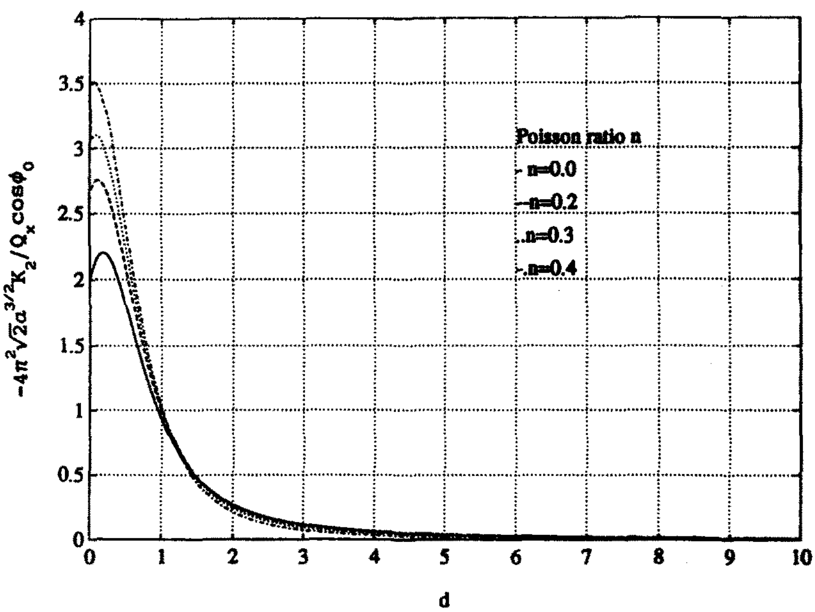

Fig. 11. Stress intensity factor K_2 due to force Q_x at a point on the normal axis.

The calculation of the maximum of the function in eqn (80) results in the same relationship as in (88). In Fig. 9 we can see that, when $\nu = 0.2$, the value of $d = 0.33$ corresponds to the maximum of K_2. Also, in Fig. 4 for the same values of ν and d at the point $\rho = 0$ the maximum of K_2 is identical to the one in Fig. 9. From Fig. 5 it is apparent that regardless of the values of ν and d, at the point $\rho = 0$, K_3 is identically zero.

The calculation of the maximum of the function in eqn (82) results in

$$d = \sqrt{\frac{1-\nu}{1+\nu}}. \tag{89}$$

Again, comparison of Figs 10 and 3 will indicate the correspondence of the maximum of K_1 for the values of, say, $\nu = 0.4$ and $d = 0.655$ as a result of (89).

Analogous conclusions could be drawn for the rest of the graphical representations.

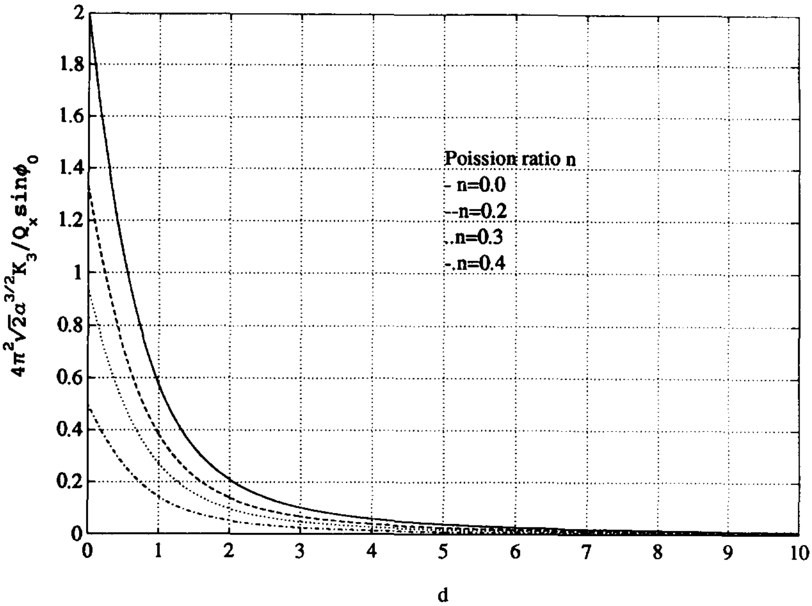

Fig. 12. Stress intensity factor K_3 due to force Q_x at a point on the normal axis.

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APPENDIX

The main limiting quantity, which needs to be computed, is

$$\lim_{\rho_0\to a}\left\{\frac{f_2(z)}{(a-\rho_0)^{1/2}}\right\} = \lim_{\rho_0\to a}\left\{\frac{1}{(a-\rho_0)^{1/2}}\frac{1}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right)\right\}.$$

Since as $\rho_0 \to a$ the argument of $\tan^{-1}$ tends to zero, we may use either L'Hôpital's rule or the fact that the $\tan^{-1}$ of a small argument is equal to the argument itself, which results in

$$\lim_{\rho_0\to a}\left\{\frac{1}{(a-\rho_0)^{1/2}}\frac{h}{R_0^2}\right\} = \lim_{\rho_0\to a}\left\{\frac{\sqrt{a^2-l_1^2}\sqrt{a^2-\rho_0^2}}{(a-\rho_0)^{1/2}R_0^2 a}\right\} = \frac{\sqrt{2}(a^2-l_1^2)^{1/2}}{R^2\sqrt{a}}.$$

Here h, R and l_1 were defined in eqns (13), (25) and (26) respectively. It is not difficult to check that the use of L'Hôpital's rule renders the same result.

The observation of the formulae in (5)–(10) reveals that they all contain either $(a^2-\rho_0^2)^{1/2}$ or h as a multiplying factor. This makes the actual computation of the rest of the limits trivial.